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CALCULATION OF THE JUMP FREQUENCIES IN THE RESPONSE OF s.d.o.f. NON-LINEAR SYSTEMS

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1. INTRODUCTION

It is a well-known fact that the non-linearity present in a system leads to jumps in the frequency– and force–response curves [1]. As shown in Figure 1, the frequency–response curve of a Duffing oscillator is bent either to the left or to the right, depending on whether the type of non-linearity is softening or hardening. The bending of the frequency–response curve leads to a jump in the response amplitude when the excitation frequency is swept from left-to-right or right-to-left. The response amplitude increases at a jump-up point and decreases at a jump-down point. Between the jump points, multiple solutions exist for a given value of the excitation frequency, and the initial conditions determine which of these solutions represents the actual response of the system. The jump points of a frequency–response curve coincide with the turning points of the curve where saddle-node bifurcations occur. The goal of this letter is to determine the minimum forcing amplitude that would lead to jumps in the frequency–response curves of single-degree-of-freedom (s.d.o.f.) non-linear systems, and to also locate the jump-up and jump-down points in the frequency–response curve when the forcing amplitude is above the minimum value.

Worden [2] and Friswell and Penny [3] computed the bifurcation points of the frequency–response curve of a Duffing oscillator with linear damping. They used the method of harmonic balance to obtain the frequency–response function. To compute the jump frequencies, Worden [2] set the discriminant of the frequency–response function, which is a cubic polynomial in the square of the amplitude, equal to zero, while Friswell and Penny [3] used a numerical approach based on Newton's method. Their first order approximation results agree well with the "exact" results. But for systems with higher order geometric, inertia, and/or damping non-linearities, a more general and simple method of determining the jump frequencies is required. In this letter, we present two methods based on the elimination theory of polynomials [4, 5], which can be used to determine both the critical forcing amplitude as well as the jump frequencies in the case of s.d.o.f.-non-linear systems. Also, the methods are devoid of convergence problems associated with bad initial guesses, and have the potential of being applicable to multiple-degree-of-freedom (m.d.o.f.) non-linear systems [6, 7]. The proposed methods are outlined in the context of a single-mode response of an externally excited cantilever beam possessing cubic geometric and inertia non-linearities and linear and quadratic damping.

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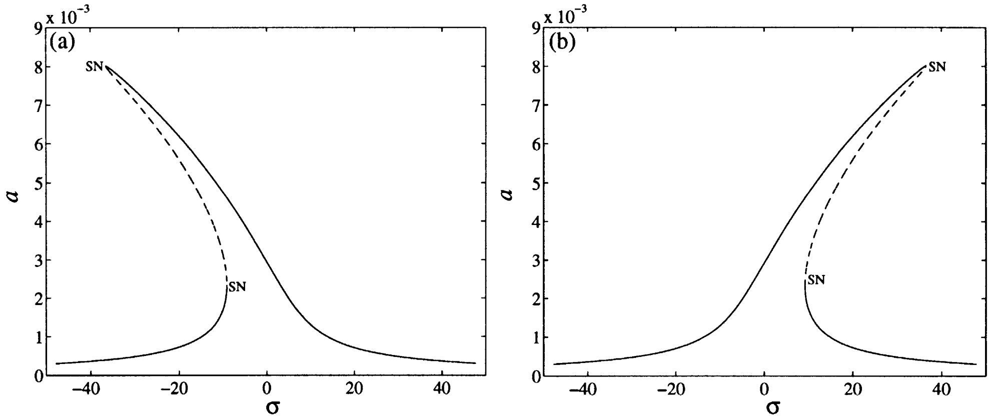

Figure 1. Typical frequency–response curves of a Duffing oscillator with (a) softening non-linearity and (b) hardening non-linearity. – – – indicate unstable solutions and SN refers to a saddle-node bifurcation.

2. THEORY

2.1. FREQUENCY–RESPONSE FUNCTION

As the cantilever beam constitutes a weakly damped, weakly non-linear system, we use the method of multiple scales [8] to derive the modulation equations governing the amplitude and phase of the excited mode of the cantilever beam. In the process of deriving the modulation equations, we define the following quantities:

$$\mu \equiv \zeta\omega_n, \quad \sigma \equiv \Omega - \omega_n, \quad f \equiv a_b \int_0^l \Phi_n \,\mathrm{d}s, \quad c \equiv \frac{4\omega_n}{3\pi}\,\bar{c} \int_0^l \Phi_n^2 |\Phi_n| \,\mathrm{d}s,$$

where ζ is the linear viscous damping factor, ω_n is the nth natural frequency of the beam, Ω is the excitation frequency, a_b is the base acceleration, l is the length of the beam, s is the arclength, $\Phi_n(s)$ is the nth mode shape, and $\bar{c}$ is the quadratic damping coefficient.

Seeking a first order uniform expansion of the transverse displacement $v(s, t)$ of the beam, we obtain

$$v(s, t) \approx a(t)\cos(\Omega t - \gamma)\Phi_n(s) + \cdots$$

and the modulation equations governing the amplitude a and phase γ of the response are given by

$$\dot{a} = -\mu a - c a^2 + \frac{f}{2\omega_n}\sin\gamma, \tag{1}$$

$$a\dot{\gamma} = \sigma a - \frac{\alpha}{4\omega_n} a^3 + \frac{f}{2\omega_n}\cos\gamma, \tag{2}$$

where α is the effective non-linearity comprising the geometric and inertia non-linearity contributions, and the overdot indicates differentiation with respect to time t. For a detailed derivation procedure of the modulation equations, we refer the reader to Anderson *et al.* [9].

Periodic solutions of the beam correspond to the fixed points of equations (1) and (2). To determine these fixed points, we set the right sides of equations (1) and (2) equal to zero. We, thus, obtain the following frequency–response function relating the response amplitude a and the excitation frequency Ω (or σ):

$$\sigma_{1,2} = \frac{\alpha}{4\omega_n} a^2 \mp \sqrt{\frac{f^2}{4\omega_n^2 a^2} - (\mu + ca)^2}, \tag{3}$$

where the subscript 1 and the " $-$ " sign refer to the left branch of the frequency–response curve, while the subscript 2 and the " $+$ " sign refer to the right branch. Equation (3) can be rewritten in polynomial form as

$$\mathscr{F}(a, \sigma) = a^6 + pa^4 + qa^3 + ra^2 + s = 0, \tag{4}$$

where

$$p = \frac{16\omega_n^2}{\alpha^2}\left(c^2 - \frac{\alpha}{2\omega_n}\sigma\right), \quad q = \frac{32\omega_n^2}{\alpha^2}\mu c, \quad r = \frac{16\omega_n^2}{\alpha^2}(\mu^2 + \sigma^2), \quad s = -\frac{4f^2}{\alpha^2}.$$

The frequency–response function can also be written as a polynomial function in σ as follows:

$$\mathscr{F}(a, \sigma) = p\sigma^2 + q\sigma + r = 0, \tag{5}$$

where

$$p = \frac{16\omega_n^2}{\alpha^2} a^2, \quad q = -\frac{8\omega_n}{\alpha} a^4, \quad r = a^6 - \frac{4f^2}{\alpha^2} + \frac{16\omega_n^2}{\alpha^2}(c^2a^4 + 2c\mu a^3 + \mu^2 a^2).$$

2.2. SYLVESTER RESULTANT

The resultant of two polynomials is defined as the product of all of the differences between the roots of the polynomials, and is a polynomial in the coefficients of the two polynomials [4]. Consider two polynomials $f(x)$ and $g(x)$ defined as

$$f(x) \equiv \sum_{i=0}^{n} a_i x^i, \quad a_n \neq 0, \qquad g(x) \equiv \sum_{i=0}^{m} b_i x^i, \quad b_m \neq 0.$$

Then, the Sylvester resultant of $f(x)$ and $g(x)$, denoted by $\mathscr{R}(f, g)$, is given by [5]:

$$\mathscr{R}(f, g) = \begin{vmatrix} a_n & a_{n-1} & \cdots & \cdots & a_1 & a_0 & 0 & \cdots & \cdots & 0 \\ 0 & a_n & a_{n-1} & \cdots & \cdots & a_1 & a_0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & & & \cdots & \cdots & \cdots & \cdots & \\ 0 & \cdots & 0 & a_n & a_{n-1} & \cdots & \cdots & \cdots & \cdots & a_0 \\ b_m & b_{m-1} & \cdots & b_1 & b_0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & b_m & b_{m-1} & \cdots & b_1 & b_0 & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & & & \cdots & \cdots & \cdots & \cdots & \\ 0 & \cdots & \cdots & 0 & b_m & b_{m-1} & \cdots & \cdots & \cdots & b_0 \end{vmatrix}.$$

A necessary and sufficient condition for $f(x)$ and $g(x)$ to have a common root is that the resultant $\mathscr{R}(f, g)$ be equal to zero [4]. The discriminant Δ of a polynomial $f(x)$, of order m, is related to the resultant $\mathscr{R}(f, f')$ in the following manner:

$$\mathscr{R}(f, f') = (-1)^{(1/2)m(m-1)} a_m \Delta,$$

where a_m is the coefficient of the x^m term in the polynomial $f(x)$. We know that $f(x) = 0$ has two equal roots if $f(x) = 0$ and $f'(x) = 0$ have a common root, and hence if $\mathscr{R}(f, f') = 0$. We use this idea to determine the critical forcing amplitude and jump frequencies.

2.3. CRITICAL FORCING AMPLITUDE

For a low excitation amplitude, we do not observe the jump phenomenon and the frequency–response curve is single valued; that is, for every value of Ω there is a unique value of a. But in the case of a large excitation amplitude, we observe jumps, and for a range of Ω values there exist multiple values of a for a given value of Ω, as seen in Figure 1. Let f_{cr} denote the critical value of f marking the boundary between the values of f leading to jumps and those not leading to jumps. The frequency–response curve for $f = f_{cr}$ has an inflection point, which we denote by (σ_{cr}, a_{cr}), where the frequency–response function $\mathscr{F}(a, \sigma_{cr}) = 0$ has three positive real roots equal to a_{cr}. Therefore, the derivative of the frequency–response function with respect to the response amplitude a, denoted by $\mathscr{F}'(a, \sigma_{cr}) = 0$, has two real roots equal to a_{cr}, which requires that the resultant $\mathscr{R}(\mathscr{F}', \mathscr{F}'')$ be equal to zero at the inflection point (σ_{cr}, a_{cr}). Thus, using equation (5), we obtain

$$\mathscr{S}(a_{cr}) \equiv \mathscr{R}(\mathscr{F}', \mathscr{F}'')|_{a=a_{cr}} = \sum_{i=0}^{6} b_i a_{cr}^i = 0, \tag{6}$$

where

$$b_0 = 144c^2\mu^2\omega_n^4, \quad b_1 = 384c^3\mu\omega_n^4, \quad b_2 = 64\omega_n^2(\alpha^2\mu^2 + 4c^4\omega_n^2),$$
$$b_3 = 168c\alpha^2\mu\omega_n^2, \quad b_4 = 96c^2\alpha^2\omega_n^2, \quad b_5 = 0, \quad b_6 = -3\alpha^4.$$

We now have a sextic polynomial equation in the response amplitude at the inflection point a_{cr}. Using the resultant, we basically eliminate σ_{cr} and obtain a polynomial equation in a_{cr} only. By using equation (4), we can eliminate a_{cr} and obtain a polynomial equation in σ_{cr}, but that would involve a more number of computations. Also, in that case spurious solutions appear while solving for σ_{cr}.

Knowing the b_i, one can easily compute the value of a_{cr} numerically. Of the six roots of $\mathscr{S}(a_{cr}) = 0$, only one turns out to be real and positive. Once we know the value of a_{cr}, substituting it into $\mathscr{F}''(a, \sigma_{cr}) = 0$ gives us the critical excitation frequency σ_{cr}. Using the values of σ_{cr} and a_{cr} in equation (4) or (5), we obtain the critical forcing amplitude f_{cr}.

For the case of linear damping ($c = 0$), a closed-form solution for the critical forcing amplitude is possible. The corresponding expressions of f_{cr}, a_{cr}, and σ_{cr} are as follows:

$$f_{cr} = 8\mu\omega_n\sqrt{\frac{2\mu\omega_n}{3\sqrt{3}|\alpha|}}, \quad a_{cr} = \sqrt{\frac{8\mu\omega_n}{\sqrt{3}|\alpha|}}, \quad \sigma_{cr} = \pm\sqrt{3}\mu,$$

where the " + " sign is for systems with effective hardening non-linearity (i.e., $\alpha > 0$), and the " − " sign is for systems with effective softening non-linearity (i.e., $\alpha < 0$).

2.4. JUMP FREQUENCIES

For $f > f_{cr}$, we observe jumps in the frequency–response curve, as seen in Figure 1. At the jump points, which we denote by (σ^*, a^*), the frequency–response function $\mathscr{F}(a, \sigma^*) = 0$ has two positive real roots equal to a^*, which requires that the resultant $\mathscr{R}(\mathscr{F}, \mathscr{F}')$ be equal to zero at those points. Using equation (5), we, thus, obtain a 12th order polynomial equation in a^* as follows:

$$\mathscr{S}(a^*) \equiv \mathscr{R}(\mathscr{F}, \mathscr{F}')|_{a=a^*} = \sum_{i=0}^{12} c_i a^i|_{a=a^*} = 0, \tag{7}$$

where the c_i are functions of known physical quantities. The values of a^* can be easily computed numerically. Of the 12 roots of $\mathscr{S}(a^*) = 0$, only two turn out to be real and positive. Once we know the value of a^*, substituting it into $\mathscr{F}'(a, \sigma^*) = 0$ gives us the jump frequency σ^*. But for each value of a^*, we obtain two values of σ^*, one of which is spurious. To pin-point the spurious σ^* solution, we check if $\mathscr{F}(a, \sigma^*) = 0$ leads to two positive real roots equal to a^*. If it does not, then that particular σ^* solution is spurious and is discarded.

2.5. GRÖBNER BASIS

A Gröbner basis for the polynomials $\{f_1, f_2, \dots, f_n\}$ comprises a set of polynomials $\{g_1, g_2, \dots, g_m\}$ that have the same collection of roots as the original polynomials [7]. Like the Sylvester resultant, the Gröbner basis also can be used to determine the critical forcing amplitude and jump frequencies. The advantage of using Gröbner bases over resultants is that we do not obtain any spurious solutions while solving for the jump frequencies σ^*. But in general, resultants are more efficient than Gröbner bases.

To determine the critical forcing amplitude, we use the fact that $\mathscr{F}'(a, \sigma) = 0$ and $\mathscr{F}''(a, \sigma) = 0$ at the inflection point (σ_{cr}, a_{cr}). We begin by computing a Gröbner basis for the polynomials $\mathscr{F}'(a, \sigma)$ and $\mathscr{F}''(a, \sigma)$, and, thus, obtain two polynomials $\mathscr{G}_1$ and $\mathscr{G}_2$, which also vanish at the inflection point (σ_{cr}, a_{cr}), and have a unique structure as we shall see later. Using equation (4) or (5) and the lex order $\sigma > a$, we obtain

$$\mathscr{G}_1(a_{cr}) = \sum_{i=0}^{6} b_i a_{cr}^i = 0, \tag{8}$$

$$\mathscr{G}_2(\sigma_{cr}, a_{cr}) = 96c\alpha\mu\omega_n^3\sigma_{cr} + \sum_{i=0}^{5} c_i a_{cr}^i = 0, \tag{9}$$

where

$$b_0 = 144c^2\mu^2\omega_n^4, \quad b_1 = 384c^3\mu\omega_n^4, \quad b_2 = 64\omega_n^2(\alpha^2\mu^2 + 4c^4\omega_n^2),$$
$$b_3 = 168c\alpha^2\mu\omega_n^2, \quad b_4 = 96c^2\alpha^2\omega_n^2, \quad b_5 = 0, \quad b_6 = -3\alpha^4$$

and

$$c_0 = 192c^3\mu\omega_n^4, \quad c_1 = 64\omega_n^2(\alpha^2\mu^2 + 4c^4\omega_n^2), \quad c_2 = 132c\alpha^2\mu\omega_n^2,$$
$$c_3 = 96c^2\alpha^2\omega_n^2, \quad c_4 = 0, \quad c_5 = -3\alpha^4.$$

Equation (8) is identical to equation (6), but now we also have an additional equation $\mathscr{G}_2(\sigma_{cr}, a_{cr}) = 0$. Once the value of a_{cr} is numerically computed, we substitute it into equation (9) to obtain the value of σ_{cr}. Like before, substituting the values of a_{cr} and σ_{cr} into equation (4) or (5) gives us the critical forcing amplitude f_{cr}.

To determine the jump frequencies σ^*, we use the fact that $\mathscr{F}(a, \sigma) = 0$ and $\mathscr{F}'(a, \sigma) = 0$ at the jump points (σ^*, a^*). We begin again by computing a Gröbner basis for the polynomials $\mathscr{F}(a, \sigma)$ and $\mathscr{F}'(a, \sigma)$, and, thus, obtain two polynomials $\mathscr{G}_1$ and $\mathscr{G}_2$, which also vanish at the jump points (σ^*, a^*). Using equation (4) or (5) and the lex order $\sigma > a$, we obtain

$$\mathscr{G}_1(a^*) = \sum_{i=0}^{12} b_i a^i|_{a=a^*} = 0,$$

$$\mathscr{G}_2(\sigma^*, a^*) = \sigma^* + \sum_{i=0}^{11} c_i a^i|_{a=a^*} = 0,$$

where the b_i and c_i are functions of known physical quantities. We solve for the values of a^* and σ^* numerically. But this time, we do not obtain any spurious solutions of σ^* because of the unique form of $\mathscr{G}_2$. In this aspect, the Gröbner basis method can be viewed as a non-linear version of the Gaussian elimination technique, which is used to solve linear polynomial equations.

3. RESULTS

Following the procedure described in the previous section, we computed the critical forcing amplitude f_{cr} and the jump frequencies σ^* in the response of the cantilever beam for a value of $f > f_{cr}$. We used the `Resultant` and `Solve` functions of MATHEMATICA [10] to calculate the resultant of two polynomials and to compute roots of polynomials, and for computing a Gröbner basis for two polynomials, we used the *GroebnerBasis* function. Identical solutions are obtained using the resultant and the Gröbner basis methods. The parameter values used in the calculations are: $\omega_n = 98\,\pi$, $\text{a} = -7 + 10^8$, $\zeta = 6 + 10^{-4}$,

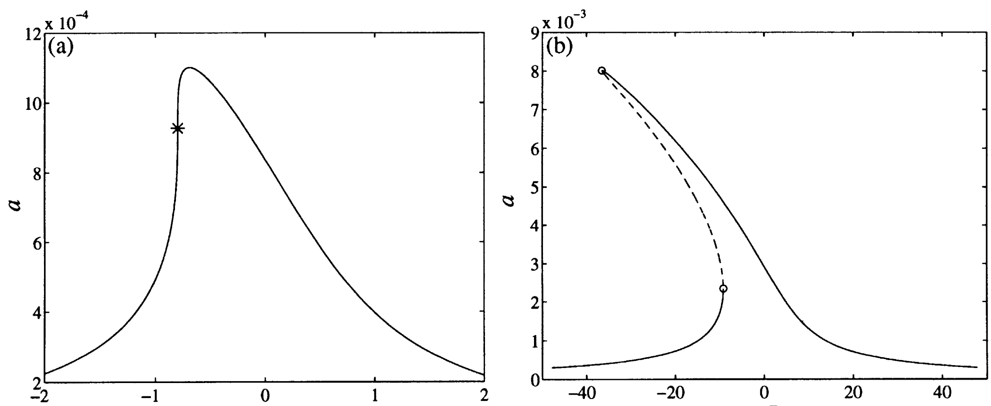

Figure 2. Frequency–response curves obtained using (a) $f = f_{cr}$ and (b) $f = 8{\cdot}82$. The asterisk in (a) indicates the inflection point and the circles in (b) indicate the jump-up and jump-down points.

$\int_0^l \Phi_n \, ds = 0{\cdot}18$ and $c = 200$. The critical forcing amplitude is found to be $f_{cr} = 0{\cdot}274$ with $\sigma_{cr} = -0{\cdot}795$ ($\Omega_{cr} = 97{\cdot}747\,\pi$) and $a_{cr} = 9{\cdot}277 \times 10^{-4}$. Using equation (3), we obtain the frequency–response curve for $f = f_{cr}$, which is illustrated in Figure 2(a). The asterisk in Figure 2(a) denotes the inflection point (σ_{cr}, a_{cr}). For $a_b = 49$ ($f = 8{\cdot}82$), the jump frequencies are found to be $\sigma^*_{up} = -9{\cdot}199$ ($\Omega^*_{up} = 95{\cdot}072\,\pi$) and $\sigma^*_{down} = -36{\cdot}544$ ($\Omega^*_{down} = 86{\cdot}368\,\pi$). The corresponding frequency–response curve is plotted, along with the computed jump-up and jump-down points, in Figure 2(b).

4. CONCLUSIONS

Knowing the form of the frequency–response function, one can easily and accurately determine the critical forcing amplitude and jump frequencies of a s.d.o.f.-non-linear system using the proposed methods. The only requirement being that the frequency–response function be a polynomial function in a and σ. The simple and straightforward methods can be applied to a variety of systems. Also, the methods have the potential of being applicable to m.d.o.f. non-linear systems.

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